

Commutator Representations of Covariant Differential Calculi on Quantum Groups

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Abstract: Let (Γ, d) be a first order differential $*$ -calculus on a $*$ -algebra \mathcal{A} . We say that a pair (π, F) of a $*$ -representation π of \mathcal{A} on a dense domain \mathcal{D} of a Hilbert space and a symmetric operator F on \mathcal{D} gives a commutator representation of Γ if there exists a linear mapping $\tau : \Gamma \rightarrow L(\mathcal{D})$ such that $\tau(adb) = \pi(a)i[F, \pi(b)]$, $a, b \in \mathcal{A}$. Among others, it is shown that each left-covariant $*$ -calculus Γ of a compact quantum group Hopf $*$ -algebra \mathcal{A} has a faithful commutator representation. For a class of bicovariant $*$ -calculi on \mathcal{A} there is a commutator representation such that F is the image of a central element of the quantum tangent space. If \mathcal{A} is the Hopf $*$ -algebra of the compact form of one of the quantum groups $SL_q(n+1)$, $O_q(n)$, $Sp_q(2n)$ with real transcendental q , then this commutator representation is faithful.

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0. Introduction

In the last decade covariant differential calculi on quantum groups have been investigated and a general theory of such calculi has been developed ([W], see [KS], Chapter 14, for a thorough treatment). Though this theory fits nicely into the general framework of quantum groups, its relations to Alain Connes' noncommutative geometry [C] are not yet understood. In [S1] we have shown that the distinguished covariant differential calculi on the quantum group $SU_q(2)$ cannot be described by a spectral triple. However they can be given by means of *unbounded* commutators in Hilbert space representations.

In this paper we address this problem in a more general setting. To be precise, suppose that Γ is a first order differential $*$ -calculus on a $*$ -algebra \mathcal{A} with differentiation $d : \mathcal{A} \rightarrow \Gamma$. Let π be a $*$ -representation of \mathcal{A} by possibly unbounded operators on a dense invariant domain \mathcal{D} of a Hilbert space and let F be a symmetric operators \mathcal{D} which leaves \mathcal{D} invariant. If there exists a well-defined linear mapping τ of Γ into the linear operators on \mathcal{D} such that

$$\tau(adb) = \pi(a)i[F, \pi(b)], a, b \in \mathcal{A}, \quad (1)$$

then the pair (π, F) is called a *commutator representation* of the $*$ -calculus Γ . We also study the problem at purely algebraic level: If π is a homomorphism of \mathcal{A} into another algebra \mathcal{B} and F is an element of \mathcal{B} such that there exist a linear map $\tau: \mathcal{A} \rightarrow \mathcal{B}$ satisfying (1), then we say that the pair (π, F) is an *algebraic commutator representation* of Γ .

We briefly state some of our main results. Each left-covariant first order calculus on a Hopf algebra \mathcal{A} admits a faithful algebraic commutator representation (Proposition 1). A large class of bicovariant calculi on coquasitriangular Hopf algebras \mathcal{A} has an algebraic commutator representation by a single element of the quantum tangent space which is central in the Hopf dual \mathcal{A}° (Proposition 4). For the coordinate Hopf algebras $\mathcal{A} = \mathcal{O}(G_q)$, $G_q = SL_q(n+1)$, $O_q(n)$, $Sp_q(2n)$, with transcendental q this representation is faithful (Theorem 5).

A crucial role in these considerations plays the cross product algebra $\mathcal{A} \rtimes \mathcal{A}^\circ$. Suppose that \mathcal{A} is a Hopf $*$ -algebra. Combined with $*$ -representations of the $*$ -algebra $\mathcal{A} \rtimes \mathcal{A}^\circ$ (more precisely, of an appropriate $*$ -subalgebra \mathcal{U}) the above algebraic commutator representations lead to Hilbert space commutator representations (Theorems 2 and 7). For instance, if \mathcal{A} is a *CQG* algebra (compact quantum group algebra), we can take the Heisenberg representation of the cross product $*$ -algebra $\mathcal{A} \rtimes \mathcal{A}^\circ$. In this manner we obtain a faithful commutator representation of each left-covariant $*$ -calculus (Corollary 3) and of bicovariant $*$ -calculi on the quantum groups $SL_q(n+1)$, $O_q(n)$ and $Sp_q(2n)$ for real transcendental q (Corollary 8).

In the final Section 4 we list three simple examples of commutator representations of covariant differential calculi on quantum spaces.

Since the FODC Γ is left-covariant with respect to \mathcal{A} , it is right-covariant with respect to \mathcal{U} . For a commutator representation of Γ one would like to have the \mathcal{U} -symmetry on the algebra $L(\mathcal{D})$ as well. For the commutator representations constructed by cross product algebras this can be done, while for the examples in Section 4 representations in larger Hilbert spaces are needed. These problems will be studied in a forthcoming paper with E. Wagner.

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1. Preliminaries

In this paper all algebras are unital and over the complex field. Algebra homomorphisms are always unit preserving. Suppose \mathcal{A} is an algebra. We denote by $M_n(\mathcal{A})$ the algebra of $n \times n$ -matrices with entries from \mathcal{A} and by \mathcal{A}_n the direct sum algebra $\mathcal{A} \oplus \dots \oplus \mathcal{A}$ (n times). The algebra of linear mappings of a vector space V is denoted by $L(V)$. A *first order differential calculus* (abbreviated, a FODC) over \mathcal{A} is an \mathcal{A} -bimodule Γ equipped with a linear mapping $d : \mathcal{A} \rightarrow \Gamma$ such that $d(ab) = a \cdot db + da \cdot b$ for $a, b \in \mathcal{A}$ and $\Gamma = \text{Lin}\{a \cdot db; a, b \in \mathcal{A}\}$.

Let $\rho : \mathcal{A} \rightarrow \mathcal{B}$ be an algebra homomorphism of \mathcal{A} into another algebra \mathcal{B} and let $C \in \mathcal{B}$. We shall say that the pair (ρ, C) is an *algebraic commutator representation* of a FODC Γ over \mathcal{A} if there exists a well-defined (!) linear mapping $\tau : \Gamma \rightarrow \mathcal{B}$ such that

$$\tau(adb) = i\rho(a)(C\rho(b) - \rho(b)C), a, b \in \mathcal{A}. \quad (2)$$

If τ is injective, then the pair (ρ, C) is called *faithful*. (The complex unit i in (2) is only included for a convenient treatment of $*$ -calculi.)

Now suppose that \mathcal{A} is a $*$ -algebra. A FODC over \mathcal{A} is called a *$*$ -calculus* if there exist a well-defined (!) involution $\gamma \rightarrow \gamma^*$ of the vector space Γ such that $(adb)^* = c^* \cdot d(b^*) \cdot a^*$ for all $a, b, c \in \mathcal{A}$. A *$*$ -representation* of \mathcal{A} is an algebra homomorphism π of \mathcal{A} into the algebra $L(\mathcal{D})$ of linear operators of a dense linear subspace \mathcal{D} of a Hilbert space such that

$$\langle \pi(a)\varphi, \psi \rangle = \langle \varphi, \pi(a^*)\psi \rangle, \varphi, \psi \in \mathcal{D}, a \in \mathcal{A}.$$

We say that a pair (π, F) of a $*$ -representation π of \mathcal{A} on the domain \mathcal{D} and a symmetric linear operator $F \in L(\mathcal{D})$ is a *commutator representation* of a $*$ -calculus Γ over \mathcal{A} if (π, F) is an algebraic commutator representation of Γ with $\mathcal{B} = L(\mathcal{D})$.

Suppose that \mathcal{A} and \mathcal{U} are Hopf algebras and $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{A} \rightarrow \mathbb{C}$ is a dual pairing of Hopf algebras. Then \mathcal{A} is a left \mathcal{U} -module algebra with left action \triangleright of \mathcal{U} on \mathcal{A} given by

$$f \triangleright a = a_{(1)} \langle f, a_{(2)} \rangle, f \in \mathcal{U}, a \in \mathcal{A},$$

where $\Delta(a) = a_{(1)} \otimes a_{(2)}$ is the Sweedler notation of the comultiplication of \mathcal{A} . The *left cross product algebra* $\mathcal{A} \rtimes \mathcal{U}$ is the vector space $\mathcal{A} \otimes \mathcal{U}$ equipped with the product defined by

$$(b \otimes f)(a \otimes g) = b(f_{(1)} \triangleright a) \otimes f_{(2)}g = \langle f_{(1)}, a_{(2)} \rangle ba_{(1)} \otimes f_{(2)}g$$

for $a, b \in \mathcal{A}$ and $f, g \in \mathcal{U}$, or equivalently $\mathcal{A} \rtimes \mathcal{U}$ is the algebra generated by the two subalgebras \mathcal{A} and \mathcal{U} with cross relation

$$fa = (f_{(1)} \triangleright a)f_{(2)} = \langle f_{(1)}, a_{(2)} \rangle a_{(1)} f_{(2)}, \quad a \in \mathcal{A}, f \in \mathcal{U}. \quad (3)$$

2. Commutator representation of left-covariant FODC on Hopf algebras

In this section Γ is a finite dimensional left-covariant FODC of a Hopf algebra \mathcal{A} . We freely use some facts on covariant FODC; see [KS], 14.1.

Let $\{\omega_1, \dots, \omega_n\}$ be a basis of the vector space of left-invariant elements of Γ . Then there exists functionals $X_k, f_l^k, k, l = 1, \dots, n$, of the dual Hopf algebra \mathcal{A}° of \mathcal{A} such that

$$da = \sum_k (X_k \triangleright a) \omega_k, \quad (4)$$

$$\omega_k a = \sum_j (f_j^k \triangleright a) \omega_j, \quad (5)$$

$$\Delta(X_k) = \varepsilon \otimes X_k + \sum_j X_j \otimes f_k^j \quad (6)$$

for $a \in \mathcal{A}$. The linear span \mathcal{T}_Γ of functionals X_1, \dots, X_n is the quantum tangent space of the left-covariant FODC Γ . From (6) and (3) it follows that the functionals X_k, f_j^k and elements $a \in \mathcal{A}$ satisfy the commutation relations

$$X_k a = a X_k + \sum_j (X_j \triangleright a) f_k^j \quad (7)$$

$$f_j^k a = \sum_l (f_l^k \triangleright a) f_j^l \quad (8)$$

in the cross product algebra $\mathcal{A} \rtimes \mathcal{A}^\circ$. There is a left action \cdot of the algebra $\mathcal{A} \rtimes \mathcal{A}^\circ$ on \mathcal{A} given by $f.b = f \triangleright b$ and $a.b = a \cdot b$ for $f \in \mathcal{A}^\circ$ and $a, b \in \mathcal{A}$. By matrix multiplication this yields a left action of the matrix algebra $M_{n+1}(\mathcal{A} \rtimes \mathcal{A}^\circ)$ on $\mathcal{A}_{n+1} = \mathcal{A} \oplus \dots \oplus \mathcal{A}$. In this manner, $M_{n+1}(\mathcal{A} \rtimes \mathcal{A}^\circ)$ becomes a subalgebra of $L(\mathcal{A}_{n+1})$. Define elements C and $\Omega_k, k = 1, \dots, n$, of $M_{n+1}(\mathcal{A}^\circ) \subseteq M_{n+1}(\mathcal{A} \rtimes \mathcal{A}^\circ) \subseteq L(\mathcal{A}_{n+1})$ by

$$C = \begin{pmatrix} 0 & X_1 & \cdots & X_n \\ X_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ X_n & 0 & \cdots & 0 \end{pmatrix}, \quad \Omega_k = i \begin{pmatrix} 0 & f_1^k & \cdots & f_n^k \\ f_1^k & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ f_n^k & 0 & \cdots & 0 \end{pmatrix}$$

Using equations (4), (5), (7) and (8) we derive that

$$i(Ca - aC).(b_0, \dots, b_n) = \sum_k (X_k \triangleright a) \Omega_k.(b_0, \dots, b_n), \quad (9)$$

$$\Omega_k a.(b_0, \dots, b_n) = \sum_l (f_l^k \triangleright a) \Omega_l.(b_0, \dots, b_n) \quad (10)$$

for $a, b_0, \dots, b_n \in \mathcal{A}$ and $k = 1, \dots, n$. Define $\varphi(a)(b_0, \dots, b_n) = (ab_0, \dots, ab_n)$. From the relations (9) and (10) we obtain

Proposition 1. *The pair (ρ, C) is a faithful algebraic commutator representation of the left-covariant FODC Γ over \mathcal{A} with $\mathcal{B} = L(\mathcal{A}_{n+1})$.*

Now we turn to Hilbert space commutator representations. We suppose that \mathcal{A} is a Hopf $*$ -algebra and Γ is a $*$ -calculus.

Then the quantum tangent space \mathcal{T}_Γ of Γ is $*$ -invariant ([KS], Proposition 14.6). Hence we can assume without loss of generality that all functionals X_i are hermitean, that is, $X_j^* = X_j$ for $j = 1, \dots, n$. Further, since \mathcal{A} is a left module $*$ -algebra for the Hopf algebra \mathcal{A}° , the cross product algebra $\mathcal{A} \rtimes \mathcal{A}^\circ$ is a $*$ -algebra which contains \mathcal{A} and \mathcal{A}° as $*$ -subalgebras ([SW], Lemma 2.1). Let \mathcal{U}_0 be a $*$ -subalgebra of \mathcal{A}° which contains all functionals $X_j, f_j^k, k, j = 1, \dots, n$. For notational simplicity we denote the $*$ -subalgebra of $\mathcal{A} \rtimes \mathcal{A}^\circ$ generated by \mathcal{A} and \mathcal{U}_0 by $\mathcal{A} \rtimes \mathcal{U}_0$.

Suppose that π_1 is a $*$ -representation of the $*$ -algebra $\mathcal{A} \rtimes \mathcal{U}_0$ on the domain \mathcal{D} . It yields a $*$ -representation π_{n+1} of the $*$ -algebra $M_{n+1}(\mathcal{A} \rtimes \mathcal{U}_0)$ on the domain $\mathcal{D}_{n+1} := \mathcal{D} \oplus \dots \oplus \mathcal{D}$ ($n+1$ times) defined by $\pi_{n+1}((x_{kj})) := ((\pi(x_{kj})))$ for $(x_{kj}) \in M_{n+1}(\mathcal{A} \rtimes \mathcal{U}_0)$. Let π denote the $*$ -representation of \mathcal{A} on \mathcal{D}_{n+1} given by $\pi(a)(\varphi_0, \dots, \varphi_n) := (\pi_0(a)\varphi_0, \dots, \pi_0(a)\varphi_n)$, $\varphi_0, \dots, \varphi_n \in \mathcal{D}$. Since $X_j^* = X_j$ for $j = 1, \dots, n$, the operator $F := \pi_{n+1}(C)$ on the domain \mathcal{D}_{n+1} is symmetric. Since π_{n+1} is an algebra homomorphism of $M_{n+1}(\mathcal{A} \rtimes \mathcal{U}_0)$ into $L(\mathcal{D}_{n+1})$, relations (9) and (10) remain valid for the corresponding images in Hilbert space. That is, we have proved

Theorem 2. *Suppose Γ is a finite dimensional left-covariant first order $*$ -calculus of the Hopf $*$ -algebra \mathcal{A} . If π_1 is a $*$ -representation of the $*$ -algebra $\mathcal{A} \rtimes \mathcal{U}_0$ defined above, then the pair (π, F) is a commutator representation of Γ . It is faithful when the $*$ -representation π_1 is faithful.*

By the preceding theorem, $*$ -representations of cross product algebras allow us to construct commutator representations of left-covariant FODC. This observation was one of the key motivations for the study of Hilbert space representations of cross product algebras in [SW].

Now suppose that \mathcal{A} is a *CQG* algebra, that is, \mathcal{A} is a Hopf $*$ -algebra which is the linear span of matrix elements of finite dimensional unitary corepresentations of \mathcal{A} (see e.g. [KS], 11.3). Let h denote the Haar state of \mathcal{A} and let π_h denote the Heisenberg representation of the cross product $*$ -algebra $\mathcal{A} \rtimes \mathcal{A}^\circ$ ([SW], 5.2). That is, π_h is the unique $*$ -representation of $\mathcal{A} \rtimes \mathcal{A}^\circ$ such that its restriction to \mathcal{A} is the GNS representation of the state h on $\pi_h(\mathcal{A})\varphi_n$ and $\pi_h(f)\varphi_h = f(1)\varphi_h$ for $f \in \mathcal{A}^\circ$. Applying Theorem 2 with $\pi_1 = \pi_h|_{\mathcal{A} \rtimes \mathcal{U}_0}$ we obtain

Corollary 3. *Each finite dimensional left-covariant first order $*$ -calculus of a CQG algebra has a faithful commutator representation.*

3. Commutator representation of bicovariant FODC on coquasitriangular Hopf algebras

In this section we suppose that \mathcal{A} is a coquasitriangular Hopf algebra with universal \mathbf{r} -form \mathbf{r} (see e.g. [KS], Chapter 10, for this notion). Let $\bar{\mathbf{r}}$ denote the convolution inverse of \mathbf{r} . First let us briefly repeat the general construction of bicovariant FODC over \mathcal{A} developped in [KS], 14.5–14.6. In a special case this method was invented in [Ju].

Fix a matrix corepresentation $v=(v_l^k)_{k,l=1,\dots,n}$ and a central character ζ of \mathcal{A} . That is, $v=(v_l^k)$ is a matrix from $M_n(\mathcal{A})$ satisfying $\Delta(v_l^k)=\sum_j v_j^k \otimes v_l^j$ and $\varepsilon(v_l^k)=\delta_{kl}$, $k, l=1, \dots, n$ and ζ is an algebra homomorphism of \mathcal{A} into \mathbb{C} .

Define functionals $l_j^{\pm k}, l_j^k \in \mathcal{A}^\circ$ by

$$l_j^{+k}(\cdot) = \mathbf{r}(\cdot \otimes v_j^k), l_j^{-k}(\cdot) = \bar{\mathbf{r}}(v_j^k \otimes \cdot), l_j^k = \sum_t S(l_t^{-k})l_j^{+t}.$$

Let $L^+, L^{-,c} \in M_n(\mathcal{A}^\circ)$ and $v^c \in M_n(\mathcal{A})$ be the matrices with (k, j) entries l_j^{+k} , $S(l_k^{-j})$ and $S(v_k^j)$, respectively. Set

$$f_{st}^{kj} := \zeta S(l_k^{-s})l_t^{+j}, k, j, s, t = 1, \dots, n. \quad (11)$$

Then, $\tilde{\Gamma}_{v,\zeta} := (1 \otimes v \otimes v^c, \zeta \otimes L^{-,c} \otimes L^+)$ is a Hopf bimodule of \mathcal{A} . On the canonical basis $\{\theta_{kj}, k, j = 1, \dots, n\}$ of left-invariant elements the \mathcal{A} -bimodule structure of $\tilde{\Gamma}_{v,\zeta}$ is determined by the relations

$$\theta_{kj}a = \sum_{s,t} (f_{st}^{kj} \triangleright a) \theta_{st}, a \in \mathcal{A}. \quad (12)$$

The element $\theta := \sum_k \theta_{kk}$ is biinvariant. Defining

$$da := \theta a - a\theta, a \in \mathcal{A}, \quad (13)$$

$\Gamma_{v,\zeta} := \mathcal{A} \cdot d\mathcal{A} \cdot \mathcal{A}$ is a bicovariant FODC over \mathcal{A} with differentiation map d . From (11) and (12) we conclude that the quantum tangent space of $\Gamma_{v,\zeta}$ is the linear span of functionals

$$X_{kj} = \zeta l_j^k - \delta_{kj} \varepsilon, k, j = 1, \dots, n. \quad (14)$$

Put $A_k^j := \sum_s \mathbf{r}(S^2(v_s^j) \otimes v_k^s)$. By Proposition 4.16(ii) in [KS], the functional

$$C_{v,\zeta} := \sum_{k,j} X_{kj} A_k^j \quad (15)$$

of the quantum tangent space of $\Gamma_{v,\zeta}$ belongs to the center of the dual Hopf algebra \mathcal{A}° .

Now we construct an algebraic commutator representation of the FODC $\Gamma_{v,\zeta}$ in the cross product algebra $\mathcal{A} \rtimes \mathcal{A}^\circ$. Set $\Omega_{kj} = \sum_{s,t} \zeta S(l_k^{-s}) l_t^{+j} A_s^t$. Since

$$\Delta(\Omega_{kj}) = \sum_{i,l,s,t} \zeta S(l_k^{-i}) l_l^{+j} \otimes \zeta S(l_i^{-s}) l_t^{+l} A_s^t = \sum_{i,l} f_{il}^{kj} \otimes \Omega_{il},$$

it follows from (3) that the elements Ω_{kj} and $a \in \mathcal{A}$ satisfy the commutation relations

$$\Omega_{kj} a = \sum_{i,l} (f_{il}^{kj} \triangleright a) \Omega_{il} \quad (16)$$

in the algebra $\mathcal{A} \rtimes \mathcal{A}^\circ$. Comparing (12) and (16), we see that there is an \mathcal{A} -bimodule map $\tilde{\tau} : \tilde{\Gamma}_{v,\zeta} \rightarrow \mathcal{A} \rtimes \mathcal{A}^\circ$ such that $\tilde{\tau}(\theta_{kj}) = \Omega_{kj}$, $k, j = 1, \dots, n$. Since $\tilde{\tau}(\theta) = C_{v,\zeta} + (Tr A) \varepsilon$ we derive

$$\tilde{\tau}(adb) = \tilde{\tau}(a(\theta b - b\theta)) = a(C_{v,\zeta} b - b C_{v,\zeta}), a, b \in \mathcal{A}.$$

Let τ be the restriction of $\tilde{\tau}$ to $\Gamma_{v,\zeta}$ and let ρ be the algebra homomorphism of \mathcal{A} into $L(\mathcal{A})$ defined by $\rho(a)b = a \cdot b$, $b \in \mathcal{A}$. By the preceding we have proved the following

Proposition 4. *The pair $(\rho, C_{v,\zeta})$ is an algebraic commutator representation of the bicovariant FODC $\Gamma_{v,\zeta}$ over \mathcal{A} with $\mathcal{B} = L(\mathcal{A})$.*

Let us specialize to the coordinate Hopf algebras $\mathcal{A} = \mathcal{O}(G_q)$, where G_q is one of the quantum groups $SL_q(n+1)$, $O_q(n)$ or $Sp_q(2n)$; see [FRT] or [KS], Chapter 9. Since the Hopf algebra $\mathcal{O}(G_q)$ is coquasitriangular ([KS], Theorem 10.9), the above considerations apply. In this case the algebraic commutator representation from Proposition 4 is even faithful. The proof relies on the paper [HS] which was essentially based on results from [JL] and [J].

Theorem 5. *Suppose that q is transcendental.*

(i) *Let G_q be $SL_q(n+1)$, $O_q(n)$ or $Sp_q(2n)$. Then the algebraic commutator representation $(\rho, C_{v,\zeta})$ of the FODC $\Gamma_{v,\zeta}$ is faithful.*

(ii) *Let $G_q = SL_q(n+1)$ or $G_q = Sp_q(2n)$. For any bicovariant finite dimensional FODC Γ over $\mathcal{O}(G_q)$ there exists a central element C of the dual Hopf algebra $\mathcal{O}(G_q)^\circ$ such that (ρ, C) is a faithful algebraic commutator representation of Γ with $\mathcal{B} = L(\mathcal{A})$.*

Proof. (i): Set $C := C_{v,\zeta}$ and $\mathcal{A} := \mathcal{O}(G_q)$. Let $\{Y_1, \dots, Y_m\}$ be a basis of the quantum tangent space $\tau_{v,\zeta}$ of the FODC $\Gamma_{v,\zeta}$. Then there are complex numbers α_{kj}^s such that $X_{kj} = \sum \alpha_{kj}^s Y_s$. Put $\Omega_s := \sum_{k,j} \alpha_{kj}^s \Omega_{kj}$. We compute

$$\Delta(C) = \sum_{k,j} \zeta l_j^k \otimes \Omega_{kj} - (Tr A) \varepsilon \otimes \varepsilon$$

and hence

$$C_{(1)} \langle C_{(2)}, a \rangle - \langle C, a \rangle \varepsilon = \sum_{k,j} \langle \Omega_{kj}, a \rangle X_{kj} = \sum_s \langle \Omega_s, a \rangle Y_s \quad (17)$$

for $a \in \mathcal{A}$. In the proof of Theorem 4.1 in [HS], it was shown that the quantum tangent space $\tau_{v,\zeta}$ is the linear span of elements $C_{(1)} \langle C_{(2)}, a \rangle - \langle C, a \rangle \varepsilon$, where $a \in \mathcal{A}$. (The latter is equivalent to the assertion proved in [HS] that the FODC $\Gamma_{v,\zeta}$ can be obtained by the method from [BM]. Note that in Theorem 4.1 in [HS] the quantum group $O_q(n)$ was excluded, but this part of the proof is valid for $O_q(n)$ as well.) Hence, by (17), the functionals $\langle \Omega_1, \dots, \Omega_m \rangle$ of \mathcal{A}° are linearly independent. We have

$$\tau\left(\sum_l a_l db_l\right) = \sum_{l,k,j} a_l (X_{kj} \triangleright b_l) \Omega_{kj} = \sum_{l,s} a_l (Y_s \triangleright b_l) \Omega_s$$

for $a_l, b_l \in \mathcal{A}$. Therefore, if $\tau(\sum_l a_l db_l) = 0$, then it follows from the definition of the cross product algebra $\mathcal{A} \rtimes \mathcal{A}^\circ$ and the linear independence of $\{\Omega_1, \dots, \Omega_m\}$ that $\sum_l a_l (Y_s \triangleright b_l) = 0$ for $s = 1, \dots, m$. This implies that $\sum_l a_l db_l = 0$. Thus,

τ is injective and (ρ, C) is faithful.

(ii): By Theorem 4.1 in [HS], the quantum tangent space \mathcal{T}_Γ of the FODC Γ is the direct sum of quantum tangent spaces $\mathcal{T}_{v_j, \zeta_j}, i = 1, \dots, k$, of FODC Γ_{v_j, ζ_j} . Setting $C := C_{v_1, \zeta_1} + \dots + C_{v_k, \zeta_k}$, the assertion follows by the repeating the reasoning of the proof of (i). \square

Next let us return to the case of general coquasitriangular Hopf algebras and consider Hilbert space commutator representations. For this we suppose that \mathcal{A} is a coquasitriangular Hopf $*$ -algebra and that the universal r -form \mathbf{r} of \mathcal{A} is real (that is, $\mathbf{r}(a \otimes b) = \mathbf{r}(b^* \otimes a^*)$ for $a, b \in \mathcal{A}$). Further, we assume that the matrix corepresentation $v = (v_k^j)$ is unitary (that is, $(v_k^j)^* = S(v_j^k)$ for $j, k = 1, \dots, n$) and that the character ζ is hermitean (that is, $\zeta(a^*) = \overline{\zeta(a)}$ for $a \in \mathcal{A}$). Then the bicovariant FODC $\Gamma_{v, \zeta}$ is a $*$ -calculus (see [KS], p. 520).

Lemma 6. *Under the above assumptions, we have $(C_{v, \zeta})^* = C_{v, \zeta}$ in the $*$ -algebra \mathcal{A}° .*

Proof. Using the well-known facts that $\mathbf{r}(S(a) \otimes S(b)) = r(a \otimes b)$ and $S \circ * \circ S \circ * = \text{id}$ we compute

$$\begin{aligned} \overline{A_k^j} &= \sum_s \overline{\mathbf{r}(S^2(v_s^j) \otimes v_k^s)} = \sum_s \mathbf{r}((v_k^s)^* \otimes (S^2(v_s^j))^*) \\ &= \sum_s \mathbf{r}(S(v_s^k) \otimes S^{-1}(S(v_s^j)^*)) \\ &= \sum_s \mathbf{r}(S^2(v_s^k) \otimes S(v_s^j)^*) = \sum_s \mathbf{r}(S^2(v_s^k) \otimes v_j^s) = A_j^k. \end{aligned}$$

Since $(l_j^k)^* = l_k^j$ by formula (47) in [KS], p. 347, the assertion follows from (14) and (15). \square

We now proceed as in the preceding section. Let \mathcal{U}_0 be a $*$ -subalgebra of \mathcal{A}° which contains all functionals $X_{kj}, f_{st}^{kj}, k, j, s, t = 1, \dots, n$, and let $\mathcal{A} \rtimes \mathcal{U}_0$ denote the $*$ -subalgebra of $\mathcal{A} \rtimes \mathcal{A}^\circ$ generated by \mathcal{A} and \mathcal{U}_0 . Retaining the above assumptions and notation, we have

Theorem 7. *Suppose that π_0 is a $*$ -representation of the $*$ -algebra $\mathcal{A} \rtimes \mathcal{U}_0$. Let $\pi := \pi_0|_{\mathcal{A}}$ and $F = \pi_0(C_{v, \zeta})$. Then the pair (π, F) is a commutator representation of the bicovariant first order $*$ -calculus $\Gamma_{v, \zeta}$.*

For $q \in \mathbb{R}$ the Hopf $*$ -algebras $\mathcal{O}(G_q)$ of the compact forms of quantum groups $G_q = SL_q(n+1), O_q(n), Sp_q(2n)$, are CQG algebras and coquasitriangular with real universal r -form. Hence the preceding considerations apply to $\mathcal{O}(G_q)$.

Recall that the Haar state h of the CQG algebra $\mathcal{O}(G_q)$ is faithful and that π_h denotes the Heisenberg representation of the cross product algebra $\mathcal{O}(G_q) \rtimes \mathcal{O}(G_q)^\circ$. Therefore, keeping the above assumption and combining Theorems 5 and 7 we obtain

Corollary 8. *Suppose that $q \in \mathbb{R}$ is transcendental.*

- (i) *Let G_q be $SL_q(n+1)$, $O_q(n)$ or $Sp_q(2n)$. Then the pair $(\pi_h, \pi_h(C_{v,\zeta}))$ is faithful commutator representation of the bicovariant $*$ -FODC $\Gamma_{v,\zeta}$.*
- (ii) *Let $G_q = SL_q(n+1)$ or $G_q = Sp_q(2n)$. For each finite dimensional bicovariant $*$ -FODC Γ there exists a central element $C \in \mathcal{O}(G_q)^\circ$ such that $(\pi_h, \pi_h(C))$ is a faithful commutator representation of Γ .*

4. Commutator Representations of FODC on Some Quantum Spaces

There are a number of distinguished covariant $*$ -calculi on quantum spaces which allow faithful commutator representations. In this final section we mention three such examples. We only state the corresponding formulas and omit the straightforward verifications.

Example 1. *Quantum disc, quantum complex plane*

Suppose that $\gamma \geq 0$ and $0 < q < 1$. Let $\mathcal{X}_{\gamma,q}$ denote the $*$ -algebra with a single generator z satisfying the relation

$$z^*z - q^2zz^* = \gamma(1 - q^2).$$

Note that $\mathcal{X}_{0,q}$ is the coordinate algebra of quantum complex plane, while $\mathcal{X}_{1,q}$ is the coordinate algebra of the quantum disc [KL]. On the $*$ -algebra $\mathcal{X}_{\gamma,q}$ there is a distinguished $*$ -calculus with bimodule structure given by

$$dz \cdot z = q^2zdz, dz \cdot z^* = q^{-2}z^*dz, dz^* \cdot z = q^2zdz^*, dz^* \cdot z^* = q^{-2}z^*dz^*.$$

These simple relations have been found in [S2]. For $\gamma \neq 0$ the FODC Γ has been extensively studied in [CHZ] and [SSV].

Define $C \in M_2(\mathcal{X}_{\gamma,q})$ and a homomorphism $\rho : \mathcal{X}_{\gamma,q} \rightarrow M_2(\mathcal{X}_{\gamma,q})$ by

$$C = (1 - q^2)^{-1} \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}, \quad \rho(f) = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$$

for $f \in \mathcal{X}_{\gamma,q}$, we have

$$[C, \rho(z)] = \begin{pmatrix} 0 & 0 \\ 1 - z^*z & 0 \end{pmatrix}, \quad [C, \rho(z^*)] = \begin{pmatrix} 0 & 1 - z^*z \\ 0 & 0 \end{pmatrix}.$$

Then the pair (ρ, C) is a faithful algebraic commutator representation of the FODC with $\mathcal{B} = M_2(\mathcal{X}_{\gamma,q})$.

Passing to a Hilbert space representation of the $*$ -algebra $\mathcal{X}_{\gamma,q}$ we get a commutator representation of the FODC. Suppose now that $\gamma = 1$. Then there is a well-known faithful $*$ -representation π of $\mathcal{X}_{\gamma,q}$ on a Hilbert space with orthonormal basis $\{e_n, n \in \mathbb{N}\}$ such that

$$\pi(z)e_n = \lambda_{n+1}e_{n+1}, \quad \pi(z^*)e_n = \lambda_n e_{n-1}, \quad \text{where } \lambda_n := (1 - q^{2n})^{1/2}.$$

In particular, $\pi(1 - z^*z)e_n = q^{2n+2}e_n$. This implies that for all $f, g \in \mathcal{X}_{\gamma,q}$ the operator $\pi(f)i[\pi(C), \pi(g)]$ is of trace class and so we have a 1-summable Fredholm module.

Note that the calculus Γ of $\mathcal{X}_{\gamma,q}$, $\gamma \neq 0$, is $\mathcal{U}_q(su(1,1))$ -covariant. There are commutator representations which have the latter symmetry, but then the commutators $[F, \pi(g)]$ are not of trace class.

Example 2. *Real quantum plane*

Suppose that q is a complex number of modulus one. Let $\mathcal{O}(\mathbb{R}_q^2)$ be the $*$ -algebra with two hermitean generators $x=x^*, y=y^*$ and defining relation $xy=qyx$. We consider $\mathcal{O}(\mathbb{R}_q^2)$ as $*$ -subalgebra of the $*$ -algebra $\hat{\mathcal{O}}(\mathbb{R}_q^2)$ with hermitean generators x, y, y^{-1} and relations $xy = qyx$ and $yy^{-1} = y^{-1}y = 1$.

There is a first order $*$ -calculus Γ of $\mathcal{O}(\mathbb{R}_q^2)$ introduced in [PW], [WZ] and given by the relations

$$xdx = q^{-2}dx \cdot x, \quad ydx = q^{-1}dx \cdot y + (q^{-2} - 1)dy \cdot x, \quad ydx = q^{-1}dx \cdot y, \quad ydy = q^{-2}dy \cdot y.$$

Let ρ be as in Example 1 and define $C \in M_2(\hat{\mathcal{O}}(\mathbb{R}_q^2))$ by

$$C = (q^2 - 1)^{-1} \begin{pmatrix} q^2 x^2 y^{-2} & 0 \\ 0 & y^{-2} \end{pmatrix}.$$

Then we have

$$[C, \rho(x)] = \begin{pmatrix} q^2 x^3 y^{-2} & 0 \\ 0 & xy^{-2} \end{pmatrix}, \quad [C, \rho(y)] = \begin{pmatrix} x^2 y^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the pair (ρ, C) is a faithful algebraic commutator representation of the FODC Γ with $\mathcal{B} = M_2(\hat{\mathcal{O}}(\mathbb{R}_q^2))$.

Composed with a faithful $*$ -representation π of $\hat{\mathcal{O}}(\mathbb{R}_q^2)$, we obtain a faithful commutator representation of Γ . For instance, we may take the $*$ -representation

π on the domain $\mathcal{D} := \text{Lin}\{e^{-\delta x^2 + cx}; \delta \geq 0, c \in \mathbb{C}\}$ of the Hilbert space $L^2(\mathbb{R})$ given by $x = e^{\alpha x}$ and $y = e^{\beta P}$, where $(Pf)(x) = -if'(x)$, $q = e^{i\gamma}$, and α and β are reals such that $\alpha\beta = \gamma$.

Example 3. *Extended quantum plane*

Suppose that $q \in \mathbb{R}$ and $q \neq 0$. Let $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ be the $*$ -algebra with generators x, y and defining relations

$$xy = qyx, xy^* = q^{-1}y^*x, x^*x = xx^*, y^*y - yy^* = (q^{-2} - 1)x^*x.$$

The $*$ -algebra $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ is the realification (see e.g. [KS], 10.2.7) of the coordinate algebra of the quantum plane \mathbb{C}_q^2 which is given by the relation $xy = qyx$. There is a first order $*$ -calculus Γ of $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ with bimodule structure described by the relations

$$\begin{aligned} dx \cdot x &= q^2 x \cdot dx, \quad dx \cdot y = qy \cdot dx + (q^2 - 1)x \cdot dy, \\ dy \cdot y &= q^2 y \cdot dy, \quad dy \cdot x = qx \cdot dy, \\ dx \cdot x^* &= q^{-2} x^* \cdot dx + (q^2 - 1)y^* \cdot dy, \quad dx \cdot y^* = q^{-1} y^* \cdot dx, \\ dy \cdot x^* &= q^{-1} x^* \cdot dy, \quad dy \cdot y^* = q^{-2} y^* \cdot dy. \end{aligned}$$

The first four relations are the same as in the preceding example. Note that the relations of this calculus are the ones described by Proposition 5 in [S3].

We first define a $*$ -representation π of $\hat{\mathcal{O}}(\mathbb{C}_q^2)$. Let \mathcal{K} be a fixed Hilbert space and let $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_n := \mathcal{K}$. For $\eta \in \mathcal{K}$ let η_n denote the vector in \mathcal{H} with n -th component η and zero otherwise. Suppose that N is a normal operator on the Hilbert space \mathcal{K} with trivial kernel. Let $N = w|N|$ be the polar decomposition of N . Since $\ker N = \{0\}$ and N is normal, w is unitary. Put $\mathcal{E} = \bigcap_{k=0}^{\infty} \mathcal{D}(|N|^k)$. Then there is a $*$ -representation π of the $*$ -algebra $\hat{\mathcal{O}}(\mathbb{R}_q^2)$ on the domain $\mathcal{D} := \text{Lin}\{\eta_n : \eta \in \mathcal{E}; n \geq 0\}$ given by

$$\begin{aligned} \pi(x)\eta_n &= q^{n+1}N\eta_n, \quad \pi(x^*)\eta_n = q^{n+1}N^*\eta_n, \\ \pi(y)\eta_n &= \lambda_n|N|\eta_{n-1}, \quad \pi(y^*)\eta_n = \lambda_{n+1}|N|\eta_{n+1}. \end{aligned}$$

Suppose that T and S are operators on the dense domain \mathcal{E} of the Hilbert space \mathcal{K} such that S is symmetric, $T(\mathcal{E}) \subseteq \mathcal{E}$, $T^*(\mathcal{E}) \subseteq \mathcal{E}$, $S(\mathcal{E}) \subseteq \mathcal{E}$ and satisfying the following conditions on the domain \mathcal{E} :

$$w^*Tw = qT, \quad w^*Sw = q^2S, \quad T|N| = |N|T, \quad S|N| = |N|S.$$

Let F denote the symmetric operator on the domain \mathcal{D} defined by

$$F\eta_n = \lambda_n T\eta_{n-1} + S\eta_n + \lambda_{n+1} T^* \eta_{n+1}, \quad \eta \in \mathcal{K}, n \in \mathbb{N}.$$

Then the pair (π, F) is a commutator representation of the $*$ -calculus Γ .

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